

# Relativistic Newton and Coulomb Laws

Yury M. Zinoviev\*

Steklov Mathematical Institute, Gubkin St. 8, 119991, Moscow, Russia,  
e - mail: zinoviev@mi.ras.ru

**Abstract.** The relativistic equations for the electromagnetic and gravitation interactions are similar: The only Lagrangian equation is the equation with Lorentz force. The potential satisfies the wave equation with the right - hand side proportional to the velocity of another particle multiplied by the delta - function concentrated at the position of another particle. If the interaction propagates at the speed of light, then the wave equation has the unique solution: the Liénard - Wiechert potential. The Maxwell equations are completely defined by the obtained relativistic Coulomb law. The Coulomb law and the Newton gravity law differ from each other only in the choice of the constants. If we choose in Coulomb law the electric charges equal to the masses and choose the interaction constant of another sign, then we get Newton gravity law. If we choose in the relativistic Coulomb law the electric charges equal to the masses and choose the interaction constant of another sign, then we get the relativistic Newton gravity law.

## 1 Introduction

The celestial mechanics is based on the gravity law discovered by Newton (1687). Cavendish (1773) proved by experiment that the force of interaction between the electric charged bodies is inversely proportional to the square of distance. This discovery was left unpublished and later was repeated by de Coulomb (1785). The electrodynamics equations were formulated by Maxwell (1873). The analysis of these equations led Lorentz [1], Poincaré [2] (this paper is the short version of the paper [4]), Einstein [3] and Minkowski [5] to the creation of the theory of relativity. The Lorentz paper [1] was based on the covariance of Maxwell equations under the Lorentz transformations. The Lorentz transformation and the Lorentz group were correctly defined by Poincaré [2], [4]. The Lorentz proof [1] of the covariance of Maxwell equations under the Lorentz group was also corrected by Poincaré [2], [4]. The idea of the relativistic Newton gravity law was proposed by Poincaré [4]:

"In the paper cited Lorentz [1] found it necessary to supplement his hypothesis in such a way that the relativity postulate could be valid for other forces in addition to the electromagnetic ones. According to his idea, because of the Lorentz transformation (and therefore because of the translational movement), all forces behave like electromagnetic (despite their origin).

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"It turned out to be necessary to consider this hypothesis more attentively and to study the changes it makes in the gravity laws in particular. First, it obviously enables us to suppose that the gravity forces propagate not instantly but at the speed of light. One could think that this is a sufficient for rejecting such a hypothesis, because Laplace has shown that this cannot occur. But, in fact, the effect of this propagation is largely balanced by some other circumstance; hence, there is no any contradiction between the law proposed and the astronomical observations.

"Is it possible to find a law satisfying the condition stated by Lorentz and at the same time reducing to the Newton law in all the cases where the velocities of the celestial bodies are small to neglect their squares (and also the products of the accelerations and the distance) compared with the square of the speed of light?"

This problem was formulated in other words in the paper [2]. The general relativity [6] was another attempt to solve the gravity problem. Einstein tried hard all his life to unify the general relativity and the electrodynamics. It seems quite natural, for the Coulomb law and the Newton gravity law differ from each other only in the choice of the constants. In this paper we find the relativistic Coulomb law by making use of the Poincaré requirements and the additional requirements. The Maxwell equations are completely defined by the obtained relativistic Coulomb law. If we choose in the relativistic Coulomb law the electric charges equal to the masses and choose the interaction constant of another sign, then we get the relativistic Newton gravity law.

## 2 Relativistic laws

We look for the relativistic Coulomb law in the following form

$$mc \frac{dt}{ds} \frac{d}{dt} \left( \frac{dt}{ds} \frac{dx^\mu}{dt} \right) + \frac{q}{c} \sum_{k=0}^N \sum_{\alpha_1, \dots, \alpha_k=0}^3 \eta^{\mu\mu} F_{\mu\alpha_1 \dots \alpha_k}(x) \frac{dt}{ds} \frac{dx^{\alpha_1}}{dt} \dots \frac{dt}{ds} \frac{dx^{\alpha_k}}{dt} = 0, \quad (2.1)$$

$$\frac{dt}{ds} = c^{-1} \left( 1 - c^{-2} \left| \frac{d\mathbf{x}}{dt} \right|^2 \right)^{-1/2} \quad (2.2)$$

where  $x^0 = ct$ ,  $\mu = 0, \dots, 3$  and the diagonal  $4 \times 4$  - matrix  $\eta^{\mu\nu} = \eta_{\mu\nu}$ ,  $\eta^{00} = -\eta^{11} = -\eta^{22} = -\eta^{33} = 1$ . The definition (2.2) implies the identities

$$\begin{aligned} \sum_{\alpha=0}^3 \eta_{\alpha\alpha} \left( \frac{dt}{ds} \frac{dx^\alpha}{dt} \right)^2 &= 1, \\ \sum_{\alpha=0}^3 \eta_{\alpha\alpha} \frac{dt}{ds} \frac{dx^\alpha}{dt} \frac{dt}{ds} \frac{d}{dt} \left( \frac{dt}{ds} \frac{dx^\alpha}{dt} \right) &= 0. \end{aligned} \quad (2.3)$$

The equation (2.1) and the second identity (2.3) imply

$$\sum_{k=0}^N \sum_{\alpha_1, \dots, \alpha_{k+1}=0}^3 F_{\alpha_1 \dots \alpha_{k+1}}(x) \frac{dt}{ds} \frac{dx^{\alpha_1}}{dt} \dots \frac{dt}{ds} \frac{dx^{\alpha_{k+1}}}{dt} = 0. \quad (2.4)$$

Let the functions  $F_{\alpha_1 \dots \alpha_{k+1}}(x)$  satisfy the equation (2.4). Then three equations (2.1) for  $\mu = 1, 2, 3$  are independent

$$m \frac{d}{dt} \left( (1 - c^{-2} |\mathbf{v}|^2)^{-1/2} v^i \right) - q \sum_{k=0}^N c^{-k} (1 - c^{-2} |\mathbf{v}|^2)^{-\frac{k-1}{2}} \times$$

$$\sum_{\alpha_1, \dots, \alpha_k=0}^3 F_{i\alpha_1 \dots \alpha_k}(x) \frac{dx^{\alpha_1}}{dt} \dots \frac{dx^{\alpha_k}}{dt} = 0, \quad v^i = \frac{dx^i}{dt}, \quad i = 1, 2, 3. \quad (2.5)$$

The following lemma is proved in the paper [7].

**Lemma 1.** *Let there exist a Lagrange function  $L(\mathbf{x}, \mathbf{v}, t)$  such that for any world line,  $x^\mu(t)$ ,  $x^0(t) = ct$ , and for any  $i = 1, 2, 3$  the relation*

$$\begin{aligned} \frac{d}{dt} \frac{\partial L}{\partial v^i} - \frac{\partial L}{\partial x^i} &= m \frac{d}{dt} \left( (1 - c^{-2} |\mathbf{v}|^2)^{-1/2} v^i \right) - q \sum_{k=0}^N \\ c^{-k} (1 - c^{-2} |\mathbf{v}|^2)^{-\frac{k-1}{2}} &\sum_{\alpha_1, \dots, \alpha_k=0}^3 F_{i\alpha_1 \dots \alpha_k}(x) \frac{dx^{\alpha_1}}{dt} \dots \frac{dx^{\alpha_k}}{dt} \end{aligned} \quad (2.6)$$

holds. Then the Lagrange function has the form

$$L(\mathbf{x}, \mathbf{v}, t) = -mc^2 (1 - c^{-2} |\mathbf{v}|^2)^{1/2} + \frac{q}{c} \sum_{i=1}^3 A_i(\mathbf{x}, t) v^i + q A_0(\mathbf{x}, t) \quad (2.7)$$

and the coefficients in the equations (2.5) are

$$F_{i\alpha_1 \dots \alpha_k}(x) = 0, \quad k \neq 1, \quad i = 1, 2, 3, \quad \alpha_1, \dots, \alpha_k = 0, \dots, 3, \quad (2.8)$$

$$\begin{aligned} F_{ij}(x) &= \frac{\partial A_j(\mathbf{x}, t)}{\partial x^i} - \frac{\partial A_i(\mathbf{x}, t)}{\partial x^j}, \\ F_{i0}(x) &= \frac{\partial A_0(\mathbf{x}, t)}{\partial x^i} - \frac{1}{c} \frac{\partial A_i(\mathbf{x}, t)}{\partial t}, \quad i, j = 1, 2, 3. \end{aligned} \quad (2.9)$$

We define

$$F_{00} = 0, \quad F_{0i} = -F_{i0}, \quad i = 1, 2, 3. \quad (2.10)$$

Then the identity

$$\sum_{\alpha, \beta=0}^3 F_{\alpha\beta}(x) \frac{dt}{ds} \frac{dx^\alpha}{dt} \frac{dt}{ds} \frac{dx^\beta}{dt} = 0 \quad (2.11)$$

similar to the identity (2.4) holds. By making use of the second identity (2.3) and the identity (2.11) we can rewrite the equation (2.5) with the coefficients (2.8), (2.9) as the equation with Lorentz force

$$mc \frac{dt}{ds} \frac{d}{dt} \left( \frac{dt}{ds} \frac{dx^\mu}{dt} \right) = -\frac{q}{c} \eta^{\mu\mu} \sum_{\nu=0}^3 F_{\mu\nu}(x) \frac{dt}{ds} \frac{dx^\nu}{dt}, \quad \mu = 0, \dots, 3. \quad (2.12)$$

Here we use the coefficients (2.9), (2.10).

The Coulomb law has the form

$$m_k \frac{d^2 x_k^i}{dt^2} = q_k \frac{\partial}{\partial x_k^i} U(\mathbf{x}_k, \mathbf{x}_j), \quad (2.13)$$

$$\sum_{i=0}^3 \left( \frac{\partial}{\partial x_k^i} \right)^2 U(\mathbf{x}_k, \mathbf{x}_j) = 4\pi q_j K \delta(\mathbf{x}_k - \mathbf{x}_j), \quad i = 1, 2, 3, \quad k, j = 1, 2, \quad k \neq j, \quad (2.14)$$

where  $q_1, q_2$  are the electric charges and  $K$  is the constant. The Newton gravity law is the equations (2.13), (2.14) with the constants  $q_k = m_k$ ,  $k = 1, 2$ ,  $K = -G$  where the gravitation constant  $G = (6.673 \pm 0.003) \cdot 10^{-11} m^3 kg^{-1} s^{-2}$ .

We define the relativistic Coulomb law

$$m_k c \frac{dt}{ds_k} \frac{d}{dt} \left( \frac{dt}{ds_k} \frac{dx_k^\mu}{dt} \right) = -\frac{q_k}{c} \eta^{\mu\nu} \sum_{\nu=0}^3 F_{\mu\nu}(x_k, x_j) \frac{dt}{ds_k} \frac{dx_k^\nu}{dt}, \quad (2.15)$$

$$F_{\mu\nu}(x_k, x_j) = \frac{\partial A_\nu(x_k, x_j)}{\partial x_k^\mu} - \frac{\partial A_\mu(x_k, x_j)}{\partial x_k^\nu}, \quad (2.16)$$

$$(\partial_{x_k}, \partial_{x_k}) A_\mu(x_k, x_j) = \eta_{\mu\mu} 4\pi q_j K \left( \frac{d}{dx_k^0} x_j^\mu (c^{-1} x_k^0) \right) \delta(\mathbf{x}_k - \mathbf{x}_j (c^{-1} x_k^0)), \quad (2.17)$$

$$(\partial_x, \partial_x) = \sum_{\nu=0}^3 \eta^{\nu\nu} \left( \frac{\partial}{\partial x^\nu} \right)^2, \quad (2.18)$$

$$\frac{dt}{ds_k} = c^{-1} \left( 1 - c^{-2} \left| \frac{d\mathbf{x}_k}{dt} \right|^2 \right)^{-1/2}, \quad \mu = 0, \dots, 3, \quad x_k^0(t) = ct, \quad j, k = 1, 2, \quad j \neq k.$$

The right - hand sides of the equations (2.14) and (2.17) differ from each other in the velocity of another particle.

The following lemma is proved in the paper [7].

**Lemma 2.** *If the world line,  $x_j^\mu(t)$ ,  $x_j^0 = ct$ , satisfies the condition*

$$\left| \frac{d\mathbf{x}_j(t)}{dt} \right| < c, \quad (2.19)$$

*then for an arbitrary matrix  $\Lambda_\nu^\mu$  from the Lorentz group*

$$\begin{aligned} & \sum_{\nu=0}^3 \Lambda_\nu^\mu \left( \frac{d}{dx_k^0} x_j^\nu (c^{-1} x_k^0) \right) \delta(\mathbf{x}_k - \mathbf{x}_j (c^{-1} x_k^0)) = \\ & c^{-1} \left( \frac{d}{dt_1} (\Lambda x_j)^\mu(t(t_1)) \right) \delta((\overrightarrow{\Lambda x_k}) - (\overrightarrow{\Lambda x_j})(t(t_1))) \Big|_{t_1 = c^{-1}(\Lambda x_k)^0} \end{aligned} \quad (2.20)$$

*where the function  $t(t_1)$  is defined by the equation*

$$ct_1 = (\Lambda x_j)^0(t(t_1)) = \sum_{\nu=0}^3 \Lambda_\nu^0 x_j^\nu(t(t_1)). \quad (2.21)$$

If the condition (2.19) for  $j = 1, 2$  is valid, the relations (2.20), (2.21) imply the Lorentz covariance of the equations (2.15) - (2.17). The Lorentz covariance of the Maxwell equations will be the consequence of the Lorentz covariance of the equations (2.15) - (2.17) and the Poincaré requirement that the interaction propagates at the speed of light.

If a distribution  $e_0(x) \in S'(\mathbf{R}^4)$  satisfies the equation

$$-(\partial_x, \partial_x) e_0(x) = \delta(x), \quad (2.22)$$

then a distribution

$$A_\mu(x_k, x_j) = -\eta_{\mu\mu} 4\pi q_j K \int d^4 y e_0(x_k - y) \left( \frac{d}{dy^0} x_j^\mu (c^{-1} y^0) \right) \delta(\mathbf{y} - \mathbf{x}_j (c^{-1} y^0)) \quad (2.23)$$

satisfies the equation (2.17). Due to the Poincaré requirement the interaction propagates at the speed of light. It means that a support of a distribution  $e_0(x)$  lies in the boundary of the upper light cone.

**Lemma 3.** *If a distribution  $e_0(x)$  has a support in the closed upper light cone and satisfies the equation (2.22), then*

$$e_0(x) = -(2\pi)^{-1}\theta(x^0)\delta((x^0)^2 - |\mathbf{x}|^2) \quad (2.24)$$

where the step function

$$\theta(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0. \end{cases} \quad (2.25)$$

**Proof.** Let the equation (2.22) have two solutions  $e^{(1)}(x)$ ,  $e^{(2)}(x)$  whose supports lie in the closed upper light cone. In view of the convolution commutativity these solutions coincide

$$e^{(2)}(x) = -(\partial_x, \partial_x) \int d^4y e^{(1)}(x-y) e^{(2)}(y) = -(\partial_x, \partial_x) \int d^4y e^{(2)}(x-y) e^{(1)}(y) = e^{(1)}(x).$$

Due to ([8], Section 30) the distribution (2.24) is the solution of the equation (2.22). The lemma is proved.

The support of the distribution (2.24) lies in the boundary of the upper light cone. The Poincaré requirement is fulfilled. The distribution (2.24) is Lorentz invariant.

The substitution of the distribution (2.24) into the equality (2.23) yields

$$A_\mu(x_k, x_j) = \eta_{\mu\mu} q_j K \left( \frac{d}{dt'} x_j^\mu(t') \right) \left( c|\mathbf{x}_k - \mathbf{x}_j(t')| - \sum_{i=1}^3 (x_k^i - x_j^i(t')) \frac{d}{dt'} x_j^i(t') \right)^{-1},$$

$$x_k^0 - ct' = |\mathbf{x}_k - \mathbf{x}_j(t')|. \quad (2.26)$$

The potentials (2.26) were introduced by Liénard (1898) and Wiechert (1900) as the generalizations of Coulomb potential.

If the velocities of bodies are small enough to neglect their squares compared with the square  $c^2$  of the speed of light and the time  $t'$  in the equality (2.26) is approximately equal to  $c^{-1}x_k^0$ , then

$$A_0(x_k, x_j) \approx q_j K |\mathbf{x}_k - \mathbf{x}_j(c^{-1}x_k^0)|^{-1},$$

$$A_i(x_k, x_j) \approx 0, \quad i = 1, 2, 3,$$

$$\frac{dt}{ds_k} \approx c^{-1}. \quad (2.27)$$

The substitution of the expressions (2.27) into the equations (2.15), (2.16) yields the Coulomb law (2.13), (2.14). All Poincaré requirements are fulfilled. If we insert the constants  $q_k = m_k$ ,  $k = 1, 2$ ,  $K = -G$  into the equations (2.15), (2.16), (2.26), then we obtain the relativistic Newton gravity law.

The interaction between  $n$  charged bodies is given by the equations

$$m_k c \frac{dt}{ds_k} \frac{d}{dt} \left( \frac{dt}{ds_k} \frac{dx_k^\mu}{dt} \right) = -\frac{q_k}{c} \eta^{\mu\mu} \sum_{\nu=0}^3 \frac{dt}{ds_k} \frac{dx_k^\nu}{dt} \sum_{j=1, \dots, n, j \neq k} F_{\mu\nu}(x_k, x_j) \quad (2.28)$$

where  $\mu = 0, \dots, 3$ ,  $k = 1, \dots, n$  and the functions  $F_{\mu\nu}(x_k, x_j)$  are given by the relations (2.16), (2.26).

**Lemma 4.** *The potentials (2.26) satisfy the equation*

$$\sum_{\mu=0}^3 \eta^{\mu\mu} \frac{\partial}{\partial x_k^\mu} A_\mu(x_k, x_j) = 0. \quad (2.29)$$

**Proof.** Let us define the vector

$$j^\mu(x_k, x_j) = 4\pi q_j K \left( \frac{d}{dx_k^0} x_j^\mu (c^{-1} x_k^0) \right) \delta(\mathbf{x}_k - \mathbf{x}_j (c^{-1} x_k^0)), \quad \mu = 0, \dots, 3. \quad (2.30)$$

The world line  $x_j^\mu(t)$  satisfies the condition  $x_j^0(t) = ct$ . Hence the definition (2.30) implies

$$\begin{aligned} \frac{\partial}{\partial x_k^0} j^0(x_k, x_j) &= -4\pi q_j K \sum_{i=1}^3 \left( \frac{d}{dx_k^0} x_j^i (c^{-1} x_k^0) \right) \frac{\partial}{\partial x_k^i} \delta(\mathbf{x}_k - \mathbf{x}_j (c^{-1} x_k^0)), \\ \frac{\partial}{\partial x_k^i} j^i(x_k, x_j) &= 4\pi q_j K \left( \frac{d}{dx_k^0} x_j^i (c^{-1} x_k^0) \right) \frac{\partial}{\partial x_k^i} \delta(\mathbf{x}_k - \mathbf{x}_j (c^{-1} x_k^0)), \quad i = 1, 2, 3. \end{aligned} \quad (2.31)$$

The relations (2.31) imply the continuity equation

$$\sum_{\mu=0}^3 \frac{\partial}{\partial x_k^\mu} j^\mu(x_k, x_j) = 0. \quad (2.32)$$

The definitions (2.23), (2.30) and the equation (2.32) imply the equality (2.29). The lemma is proved.

**Lemma 5.** *If the functions  $F_{\mu\nu}(x_k, x_j)$  are defined by the relations (2.16), (2.26), then*

$$\sum_{\mu=0}^3 \eta^{\mu\mu} \frac{\partial}{\partial x_k^\mu} \sum_{j=1, \dots, n, j \neq k} F_{\mu\nu}(x_k, x_j) = \eta_{\nu\nu} \sum_{j=1, \dots, n, j \neq k} j^\nu(x_k, x_j). \quad (2.33)$$

**Proof.** The definition (2.16) and the equation (2.29) imply

$$\sum_{\mu=0}^3 \eta^{\mu\mu} \frac{\partial}{\partial x_k^\mu} F_{\mu\nu}(x_k, x_j) = \sum_{\mu=0}^3 \eta^{\mu\mu} \left( \frac{\partial}{\partial x_k^\mu} \right)^2 A_\nu(x_k, x_j). \quad (2.34)$$

The potential  $A_\nu(x_k, x_j)$  is defined by the relations (2.23), (2.24). The distribution (2.24) satisfies the equation (2.22). Hence the equalities (2.18), (2.23), (2.30), (2.34) imply

$$\sum_{\mu=0}^3 \eta^{\mu\mu} \frac{\partial}{\partial x_k^\mu} F_{\mu\nu}(x_k, x_j) = \eta_{\nu\nu} j^\nu(x_k, x_j). \quad (2.35)$$

By summing up the equalities (2.35) we obtain the equality (2.33). The lemma is proved.

By making use of the definition (2.16) it is easy to verify for different  $\mu, \nu, \lambda$

$$\frac{\partial}{\partial x_k^\lambda} F_{\mu\nu}(x_k, x_j) + \frac{\partial}{\partial x_k^\mu} F_{\nu\lambda}(x_k, x_j) + \frac{\partial}{\partial x_k^\nu} F_{\lambda\mu}(x_k, x_j) = 0. \quad (2.36)$$

The relations (2.33), (2.36) imply that the fields in the relativistic Coulomb law (2.28) satisfy the Maxwell equations with the current given by the right - hand side of the equality (2.33). Therefore the electrodynamics is completely defined by the relativistic Coulomb law (2.16),

(2.26), (2.28). The substitution  $q_j = m_j$ ,  $K = -G$  in these equations yields the relativistic Newton gravity law.

Let us consider the equations (2.15), (2.16), (2.26) for the case  $q_1 = m_1 = 0$ ,  $q_2 = m_2$ ,  $K = -G$ . Then the second body moves freely. These equations were solved in the paper [7]. The solution is similar to that of Kepler problem. It is a principal result. There is no similar result in the general relativity. It seems that the mass of Mercury is small and the Sun moves freely. It is possible to calculate the advance of Mercury's perihelion for a hundred years. According to [7], it turns out to be  $7''.18$ . According to [6] (Chap. 40, Sec. 40.5, Appendix 40.3), the whole observed advance is  $5599''.74 \pm 0''.41$  for a hundred years. It is possible to calculate the advance of Mercury's perihelion caused by non - inertial system connected with the Earth using the Newton gravity law. According to [6] (Chap. 40, Sec. 40.5, Appendix 40.3), it turns out to be  $5025''.645 \pm 0''.50$  for a hundred years. It is also possible to calculate the advance of Mercury's perihelion caused by the gravity of other planets using the Newton gravity law. According to [6] (Chap. 40, Sec. 40.5, Appendix 40.3), it turns out to be  $531''.54 \pm 0''.68$  for a hundred years. There exists a contradiction between the remaining advance of Mercury's perihelion of  $42''.56$  for a hundred years and the calculated advance  $7''.18$ . It seems that the experimental testing of the relativistic Newton gravity law (2.16), (2.26), (2.28),  $q_j = m_j$ ,  $K = -G$  will be possible only when the entire advance of Mercury's perihelion for a hundred years is calculated using these equations. The quantum effects may be also important.

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